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THE PROBLEM OF THE CENTRE OF PERCUSSION†

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The conditions for a centre of percussion to exist in the case of a solid with a fixed axis are obtained in a new form. This enables the positions of the axis of revolution as a function of the central inertial tensor to be analysed.

The concept of a centre of percussion is associated with the case, which is important in practice, of the impulsive motion of a solid with a fixed axis when the force of the impact does not load the fixing axis, for example, when the rebound accompanying a blow by a hammer or a tennis racket is insignificant.

1. FORMULATION OF THE PROBLEM

The construction of a centre of percussion is a well-known classical problem of the dynamics of a solid. Its solution can be found in many monographs and textbooks on dynamics. It turns out that a centre of percussion does not exist in a considerable number of cases. The well-known classical criterion has the following form.

1. A fixed axis OO' is the principal axis of inertia for one of its points O''.

2. The point C lies in a plane containing the axis of fixing and the centre of mass G, and its projection onto the axis coincides with the point O''.

3. The radius of inertia of the body ρ with respect to the OO-axis is the geometric mean of the distances from points G and C to this axis. In this case, points G and C lie on one side of the axis (Fig. 1).

Every impact force, the line of action which passes through point C orthogonal to the GOO' plane, imparts a revolution to the body about the axis without impulsive reactions at the support points O and O'.

Verification of condition 1 in the case when the fixing axis is specified does not present any fundamental difficulties. However, it does not enable one to investigate the structure of the set of all permissible axes of spontaneous revolution for a given body. The aim of this paper is to obtain a criterion for the existence of a centre of percussion in a form which is more convenient for analysis.

2. A GEOMETRICAL CRITERION FOR THE EXISTENCE OF A CENTRE OF PERCUSSION

Let an impulse I act along a line which intersects the plane GOO' at a certain point C. The general impact equation, when account is taken of the fact that the reaction of the axis is zero, is [1, 2]

$$M\Delta\mathbf{V} = \mathbf{I}, \quad \mathbf{J}\Delta\mathbf{W} = \mathbf{r} \times \mathbf{I} \tag{2.1}$$

where M is the mass of the body, $\mathbf{J}$ is the central inertial tensor, $\mathbf{V}$ is the velocity of the centre of mass, $\mathbf{W}$ is the angular velocity and $\mathbf{r} = GC$. The conditions for the clamping points O and O' to be fixed have the form

$$\Delta\mathbf{V}_O = \mathbf{0}, \quad \Delta\mathbf{V}_{O'} = \mathbf{0}, \quad \mathbf{V}_O = \mathbf{V} + \mathbf{W} \times GO, \quad \Delta\mathbf{V}_{O'} = \mathbf{V} + \mathbf{W} \times GO' \tag{2.2}$$

On substituting (2.1) into Eqs (2.2), we obtain

$$\begin{aligned} &M^{-1}\mathbf{I} + \mathbf{J}^{-1}(\mathbf{r} \times \mathbf{I}) \times \mathbf{r}_O = \mathbf{0}, \quad M^{-1}\mathbf{I} + \mathbf{J}^{-1}(\mathbf{r} \times \mathbf{I}) \times \mathbf{r}_{O'} = \mathbf{0} \\ &\mathbf{r}_O = GO, \quad \mathbf{r}_{O'} = GO' \end{aligned} \tag{2.3}$$

Subtracting one of the relationships (2.3) from the other, we obtain the equality

$$\mathbf{J}^{-1}(\mathbf{r} \times \mathbf{I}) \times \mathbf{e} = \mathbf{0} \tag{2.4}$$

($\mathbf{e}$ is the direction unit vector of the OO'-axis) which means that the vectors $\mathbf{r} \times \mathbf{I}$ and $\mathbf{Je}$ are collinear, that is, for a certain real number λ

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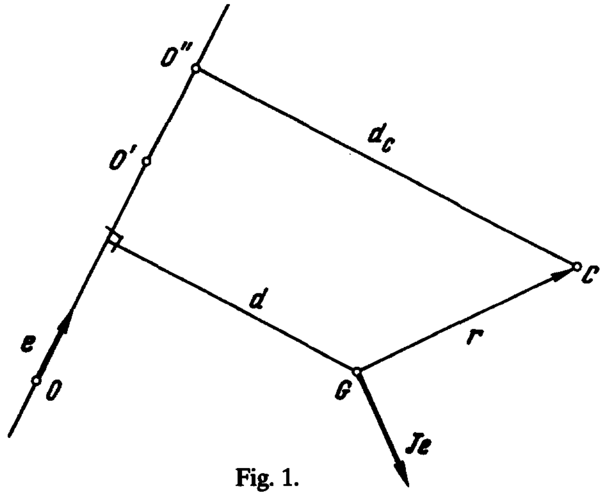

Fig. 1.

$$\mathbf{r} \times \mathbf{I} = \lambda \mathbf{J}\mathbf{e} \tag{2.5}$$

Let us substitute relationship (2.5) into the first of Eqs (2.3)

$$\mathbf{I} = -\lambda M \mathbf{e} \times \mathbf{r}_O \tag{2.6}$$

The orthogonality of the percussive impulse to the GOO'-plane follows from this.
Next, substitution of (2.6) into Eq. (2.5) transforms the latter to the form

$$\mathbf{J}\mathbf{e} = -M\mathbf{r} \times (\mathbf{e} \times \mathbf{r}_O) \tag{2.7}$$

Condition (2.7) imposes constraints on both the direction of the axis of revolution as well as on the position of the centre of impact C.

Firstly, using the properties of a double vector product, we obtain the following assertion (the geometric criterion): the vector **Je** lies in a plane containing the axis of suspension and the centre of mass of the body

$$(\mathbf{J}\mathbf{e} \times \mathbf{e}, \mathbf{r}_O) = 0 \tag{2.8}$$

Secondly, the vectors **r** and **Je** are orthogonal

$$(\mathbf{r}, \mathbf{J}\mathbf{e}) = 0 \tag{2.9}$$

Thirdly, multiplying (2.7) scalarly by **e** and using the properties of moments of inertia, we obtain

$$M\rho^2 = (\mathbf{J}\mathbf{e}, \mathbf{e}) + Md^2 = M(\mathbf{r}_O \times \mathbf{e}, \mathbf{r}_O \times \mathbf{e}) - M(\mathbf{r}_O \times \mathbf{e}, \mathbf{r} \times \mathbf{e}) = Mdd_C \tag{2.10}$$

where d and d_c are the distances from the OO'-axis to points G and C respectively.

It can be shown by direct verification that the two groups of requirements 1–3 and (2.8)–(2.10) are equivalent. While, unlike 1, criterion (2.8) uses only central moments of inertia which makes the verification simpler.

3. ANALYSIS OF THE STRUCTURE OF THE SET OF AXES OF SPONTANEOUS REVOLUTION

If we apply a percussive impulsive to a body, in the general case it will execute a spiral motion. Under certain conditions, this motion will be purely rotational. In this case, it is said that an axis of spontaneous revolution exists [1]. Obviously, such axes and only such axes possess a centre of percussion.

Let us ascertain how the set of all axes of spontaneous revolution is constructed for a given body. To do this, we introduce a system of coordinates, directing its axes along the principal axes of the central ellipsoid of inertia. If (x, y, z) are the coordinates of the point O, (e_1, e_2, e_3) are the coordinates of the vector **e** and A_1, A_2, A_3 are the principal central moments of inertia, Eq. (2.8) takes the form

$$(A_1 - A_2)e_1e_2z + (A_3 - A_1)e_1e_3y + (A_2 - A_3)e_2e_3x = 0 \tag{3.1}$$

The set of solutions of this equation has a different structure depending on the presence or absence of dynamic symmetry.

In the case of spherical symmetry $A_1 = A_2 = A_3$ and equality (3.1) is an identity. Consequently, a centre of percussion exists for any fixed axis (if the axis passes through the centre of mass, point C is removed to infinity). This result supplements the well-known conclusion of Lyapunov that every impact on a spherically symmetric body imparts a revolution to it about a certain axis.

For a solid of revolution $A_1 = A_2 \neq A_3$, Eq. (3.1) reduces to the form

$$e_1 e_3 y = e_2 e_3 x \tag{3.2}$$

Here, there are two groups of revolutions: either $e_3 = 0$, that is, the axes of revolution and symmetry are orthogonal, or $e_1 y = e_2 x$, that is, these two axes have a common point.

Finally, in the general case of a triaxial ellipsoid of inertia, a plane of possible positions of the point O corresponds to each direction of $\mathbf{e}$.

In view of the identity

$$(A_1 - A_2)e_1 e_2 e_3 + (A_3 - A_1)e_1 e_3 e_2 + (A_2 - A_3)e_2 e_3 e_1 \equiv 0$$

this plane is parallel to the vector $\mathbf{e}$ and contains the OO'-axis. A planar pencil of parallel axes of spontaneous revolution therefore corresponds to each vector $\mathbf{e}$.

On the other hand, if the point O is fixed, Eq. (3.1) will define a cone of possible directions of the axis of revolution.

For a solid of revolution, this cone is described by Eq. (3.2), that is, it decomposes into a pair of planes passing through the point O. One of these planes is perpendicular to the axis of symmetry and the second contains this axis.

For an asymmetric body, the cone is of the second order when $xyz \neq 0$, that is, the point O does not lie on any of the principal planes of the central ellipsoid of inertia. In this case, $z = 0, xy \neq 0$ (the point O lies in the principal plane but not on the principal axis), the surface (3.1) decomposes into a pair of planes: $e_3 = 0$ (the axis of revolution lies in the same principal plane of the ellipsoid of inertia) and

$$(A_3 - A_1)e_1 y + (A_2 - A_3)e_2 x = 0 \tag{3.3}$$

The plane (3.3) is orthogonal to the principal plane of inertia, containing the point O, but, unlike in the case of a solid of revolution, does not pass through the principal axis of inertia.

If, however, the point O lies on the principal axis of the central ellipsoid of inertia, $x = y = 0$, $z \neq 0$, then Eq. (3.1) describes a pair of the principal planes of the ellipsoid of inertia containing the axis.

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